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International Journal of Solids and Structures 43 (2006) 5168–5181
INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES
www.elsevier.com/locate/ijsolstr

Buckling analysis of fibers in composite materials by wave propagation analogy

Jacob Aboudi [a,*], Rivka Gilat [b]

[a] Department of Solid Mechanics, Faculty of Engineering, Tel Aviv University, Ramat Aviv 69978, Israel
[b] Department of Civil Engineering, Faculty of Engineering, The College JAS, Ariel 44837, Israel

Received 18 April 2005; received in revised form 1 June 2005
Available online 27 July 2005

Abstract

The analogy between the governing equations for the analysis of buckling in elastic structures and the elastodynamic equations of motion for wave propagation is presented. By employing this analogy, the exact and approximate buckling stresses of periodic layered materials and continuous fiber composites, respectively, are established. This is performed by utilizing micromechanically based dispersion relations for elastic wave propagating in the composite materials, which provide for a given wave length the corresponding phase velocity. By a specific change of variables in these dispersion relations, the corresponding buckling stresses can be determined. Results are presented and compared with solutions based on the mechanics of materials approach as well as with the well known Rosen's fiber buckling predictions.

Keywords: Composite materials; Fibers microbuckling; Dispersion relations; Micromechanics

1. Introduction

The buckling of fibers in elastic composite materials has been investigated by several authors. In the classical investigation of Rosen (1964), which has been summarized by Jones (1975), the buckling of fibers was analyzed by considering the two-dimensional problem of a periodically two-layered composite in which the fibers and matrix were represented by the stiff and soft layers, respectively. Rosen (1964) considered two types of buckling modes: shear and transverse buckling modes. In the first type of buckling the fiber and matrix layers exhibit in-phase deformation, whereas in the latter type the fiber and matrix layers exhibit

* Corresponding author. Tel.: +97 236408131; fax: +97 236407617.
E-mail address: aboudi@eng.tau.ac.il (J. Aboudi).

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2005.06.024

anti-phase deformation. By applying energy considerations, Rosen (1964) established that the stress buckling in shear mode of very long wave length (compared to the thickness of the stiff fiber layer) is given by $\mu_m/(1-v_f)$ where μ_m is the shear modulus of the matrix layer and v_f is the volume fraction of the fiber layer. For the transverse buckling modes, Rosen obtained the buckling stress $2v_f[v_f E_f E_m/3(1-v_f)]^{1/2}$, where E_f and E_m are the Young's moduli of the fiber and matrix layers.

A recent investigation of the buckling of fibers in elastic composite materials has been published by Parnes and Chiskis (2002). They modeled the composite as a periodic two-layered material and analyzed the problem by employing a mechanics of materials approach based on Euler–Bernoulli theory of an infinite fiber layer embedded in an elastic foundation matrix. The interaction between the fiber and matrix layers was deduced on the basis of elasticity equations. These authors provided a comprehensive list of references to various investigations of the present subject.

In the present investigation, the analogy between the equations that govern buckling of structures and elastic wave propagation in solids is utilized to determine the buckling loads of fibers in composite materials. It turns out that in the framework of the analysis of elastic wave propagation in structural composites, if a dispersion relation, which provides for a given wave length the corresponding phase velocity, can be established, it is possible to obtain the buckling load of the structure by a simple change of variables. For periodic layered composites it is possible, by employing the elastodynamic equations of motion, to establish the corresponding dispersion curves in an exact manner. By employing the wave-buckling analogy, the exact critical buckling loads of the periodic layered composite can be readily derived. For periodic continuous fiber composites, exact dispersion relations are not available. There are on the other hand several micromechanically established approximate dispersion relations which can be employed to establish the buckling stress of the composite. It should be noted that by employing a micromechanical analysis, one can establish the dispersion relation for a propagating wave in the composite from the known material properties of the constituents, their volume fraction and their detailed interaction.

This paper is organized as follows. In Section 2, the analogy between the governing equations of elastic waves and buckling is discussed. This analogy is employed in Section 3 to establish the exact buckling loads in shear and transverse modes in periodic layered composites. In Section 4, an approximate dispersion relation of continuous fiber composites which was micromechanically established by Achenbach (1976) and whose validity has been verified, is employed to obtain the corresponding buckling loads in shear mode. Possible extensions of the present approach, including the determination of the buckling loads in piezoelectric composites, are discussed in the last section.

2. Buckling-wave propagation analogy

The elastodynamic equations of motion are given by

$$\sigma_{jm,j} = \rho \frac{d^2 u_m}{dt^2} \quad j, m = 1, 2, 3 \tag{1}$$

where σ_{jm}, u_m are the components of the stress tensor and displacement vector, respectively, ρ is the mass density of the material and t is the time.

When investigating the propagation of harmonic waves in the x_1-direction, for example, in the material it is assumed that any field variable A has the form:

$$A = \overline{A} \exp[i(kx_1 - \omega t)] \tag{2}$$

where $\overline{A}$ is an amplitude factor, k is the wave number, ω is the circular frequency and i is the imaginary unit. By substituting Eq. (2) in (1) it follows that the derivative with respect to x_1 should be replaces by ik while the derivative with respect to the time t is replaced by $-\mathrm{i}\omega$. Consequently, the following equations are established:

$$\begin{aligned} &\mathrm{i}k\sigma_{11} + \sigma_{21,2} + \sigma_{31,3} + \rho\omega^2 u_1 = 0 \\ &\mathrm{i}k\sigma_{12} + \sigma_{22,2} + \sigma_{32,3} + \rho\omega^2 u_2 = 0 \\ &\mathrm{i}k\sigma_{13} + \sigma_{23,2} + \sigma_{33,3} + \rho\omega^2 u_3 = 0 \end{aligned} \tag{3}$$

Let us consider next the nonlinear equilibrium equations in terms of the second Piola–Kirchhoff (symmetric) stress tensor $\tilde{\sigma}_{jm}$. They are given (e.g., Malvern, 1969; Whitney, 1987) by

$$[\tilde{\sigma}_{jn}(\delta_{mn} + u_{m,n})]_{,j} = 0 \quad j,m,n = 1,2,3 \tag{4}$$

where δ_{mn} is the Kronecker delta.

Let us linearize Eq. (4) while assuming that an initial compressive stress σ^0_{11} is acting in the x_1-direction, for example. The resulting equations take the form

$$\sigma_{jm,j} - \sigma^0_{11} u_{m,11} = 0 \quad j,m = 1,2,3 \tag{5}$$

where $\tilde{\sigma}_{jm}$ can be written as σ_{jm} due to the linearization.

In analyzing the buckling of a structure in which an initial stress σ^0_{11} is acting in the x_1-direction, for example, every field variable B is assumed to possess the form

$$B = \overline{B}\exp[\mathrm{i}kx_1] \tag{6}$$

where $\overline{B}$ is an amplitude factor and k is the wave number of the buckled shape. By substituting Eq. (6) in (5) the derivative with respect to x_1 is replaced by ik. Accordingly, Eq. (5) takes the form

$$\begin{aligned} &\mathrm{i}k\sigma_{11} + \sigma_{21,2} + \sigma_{31,3} + \sigma^0_{11}k^2 u_1 = 0 \\ &\mathrm{i}k\sigma_{12} + \sigma_{22,2} + \sigma_{32,3} + \sigma^0_{11}k^2 u_2 = 0 \\ &\mathrm{i}k\sigma_{13} + \sigma_{23,2} + \sigma_{33,3} + \sigma^0_{11}k^2 u_3 = 0 \end{aligned} \tag{7}$$

By a comparison of Eq. (3) that governs the propagation of harmonic waves in the x_1-direction in elastic materials with Eq. (7) that has been established for the analysis of buckling of a structure with initial stress σ^0_{11} acting in the x_1-direction, one obtains that the following formal replacement holds:

$$\rho\omega^2 \Longleftrightarrow \sigma^0_{11}k^2 \tag{8}$$

This replacement implies that there is an analogy between the analyses of the buckling and elastic wave propagation in a structure. For example, if the dispersion equation (which provides a relation between the circular frequency ω and the wave number k) for wave propagation in the direction of the layering of a laminated composite can be established, one can obtain the buckling load of this composite, compressed in the direction of the layering, by replacing $\rho\,\omega^2$ of the layer by the stress in this layer multiplied by k^2. Similarly, if the dispersion equation for waves propagating in the direction of the fibers in a composite material can be derived, one can immediately employ this analogy for the determination of the buckling load of these fibers caused by a compressive loading of the composite. In the following two sections, this analogy will be utilized for the determination of the buckling load of layered and fibrous composites.

3. Exact buckling loads of periodically layered composites by wave propagation analogy

Fig. 1 shows a layered composite that is subjected to an externally applied compressive stress loadings whose average is $\overline{\sigma}_{11}$ (which is the average of the applied compressive stress loadings $\sigma^{(\mathrm{f})}_{11}$ and $\sigma^{(\mathrm{m})}_{11}$ that are

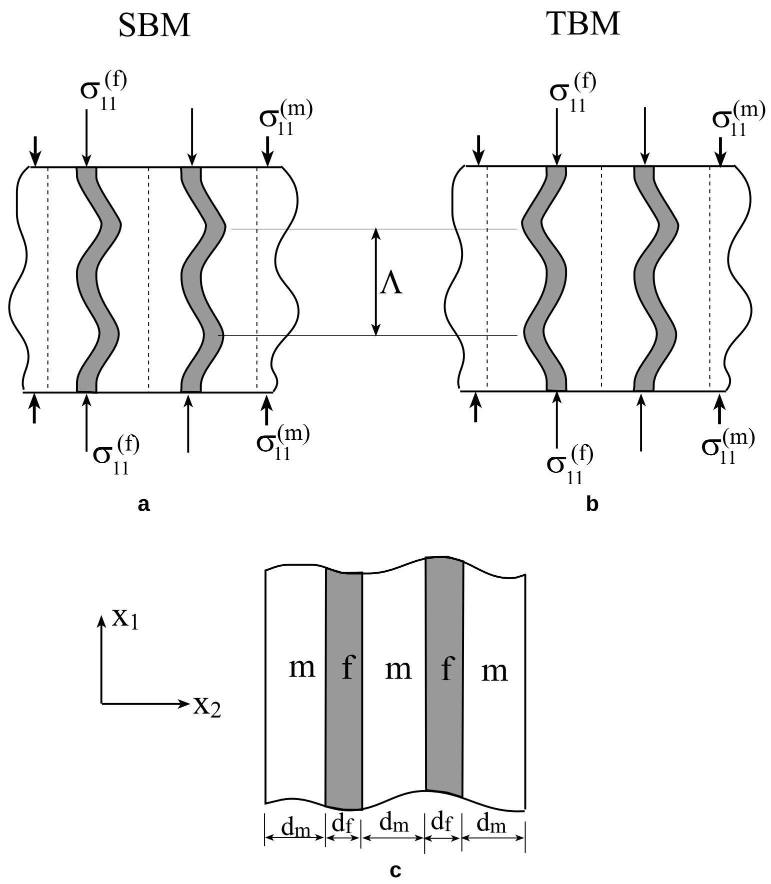

Fig. 1. (a) Buckling of fibers in shear buckling mode in a periodically layered composite. (b) Buckling of fibers in transverse buckling mode in a periodically layered composite. (c) The repeating unit cell of the periodically layered composite consists of five layers the analysis of which enables the modeling of both shear and transverse buckling modes.

imposed on the fiber and matrix layers). As shown in this figure, the composite can buckle in two different modes. In the first mode, referred to as shear buckling mode (SBM), the fiber (the stiff layer) and matrix (the soft layer) exhibit the same deformation shape (in-phase deformation). In the second mode, referred to as transverse buckling mode (TBM), the fiber and matrix exhibit anti-phase deformation. As can be observed from Fig. 1(a), in order to predict the critical buckling load in SBM and the associated wave length $\Lambda = 2\pi/k$, it is necessary to analyze a periodically bilaminated composite. Fig. 1(b) shows on the other hand that the prediction of the critical buckling load in TBM and the associated wave length, one has to consider a periodically layered composite that consists of five different layers which repeat themselves. The first, third and fifth layers are occupied by the matrix material, while the second and forth layers must be occupied by the fibers.

The prediction of the two types of buckling of the periodically layered material configuration is based on the analysis of elastic wave propagation in the direction of the layering in a periodically layered composite in which the repeating unit cell consists of five layers, Fig. 1(c). Once the dispersion relation for such a composite is established, the wave-buckling analogy given by Eq. (8) can be utilized for the determination of the critical buckling loads and the associated wave lengths in both modes.

The displacement vector $\mathbf{u}^{(\alpha)} = [u_1^{(\alpha)}(x_1, x_2, t), u_2^{(\alpha)}(x_1, x_2, t)]$ in layer α ($\alpha = 1, \ldots, 5$), due to the wave propagation in the periodically five-layered material of Fig. 1(c) is decomposed as follows:

$$\mathbf{u}^{(\alpha)} = \nabla\phi^{(\alpha)} + \nabla \times \boldsymbol{\psi}^{(\alpha)} \tag{9}$$

where the potentials $\phi^{(\alpha)}(x_1, x_2, t)$ and $\boldsymbol{\psi}^{(\alpha)}(x_1, x_2, t)$ satisfy the equations

$$\nabla^2\phi^{(\alpha)} = \frac{\rho_\alpha}{\lambda_\alpha + 2\mu_\alpha}\frac{d^2}{dt^2}\phi^{(\alpha)} \tag{10}$$

$$\nabla^2\boldsymbol{\psi}^{(\alpha)} = \frac{\rho_\alpha}{\mu_\alpha}\frac{d^2}{dt^2}\boldsymbol{\psi}^{(\alpha)} \tag{11}$$

where λ_α and μ_α are the Lame' constants of the material in the layer α. The potential vector $\boldsymbol{\psi}^{(\alpha)}$ consists of just one component which we denote henceforth by $\psi^{(\alpha)}$.

For propagating waves in the layering (x_1) direction, we can represent $\phi^{(\alpha)}$ and $\psi^{(\alpha)}$ in the form given by Eq. (2). This reduces Eqs. (10) and (11) to

$$\frac{\partial^2\phi^{(\alpha)}}{\partial x_2^2} + \xi_\alpha^2\phi^{(\alpha)} = 0 \tag{12}$$

$$\frac{\partial^2\psi^{(\alpha)}}{\partial x_2^2} + \eta_\alpha^2\psi^{(\alpha)} = 0 \tag{13}$$

where

$$\xi_\alpha^2 = \frac{\rho_\alpha\omega^2}{\lambda_\alpha + 2\mu_\alpha} - k^2, \quad \eta_\alpha^2 = \frac{\rho_\alpha\omega^2}{\mu_\alpha} - k^2$$

Consequently,

$$\phi^{(\alpha)} = [A_1^{(\alpha)}\cos(\xi_\alpha x_2) + A_2^{(\alpha)}\sin(\xi_\alpha x_2)]\exp[i(kx_1 - \omega t)] \tag{14}$$

and

$$\psi^{(\alpha)} = [A_3^{(\alpha)}\cos(\eta_\alpha x_2) + A_4^{(\alpha)}\sin(\eta_\alpha x_2)]\exp[i(kx_1 - \omega t)] \tag{15}$$

which result in 20 unknown coefficients $A_1^{(\alpha)}, \ldots, A_4^{(\alpha)}$. The continuity of the displacements and tractions at the interfaces between all layers including the interface between layer $\alpha = 5$ and $\alpha = 1$ (which enforces the periodicity conditions) yield 20 equations which provides a determinant of order 20 for the exact dispersion relation for waves in propagating in the 1-direction. The exact buckling loads in SBM and TBM are obtained by employing Eq. (8) which takes in the present case the form:

$$\rho_\alpha\omega^2 \Longleftrightarrow \sigma_{11}^{0(\alpha)}k^2 \tag{16}$$

where the axial stress $\sigma_{11}^{0(\alpha)}$ in the phase is determined from

$$\sigma_{11}^{0(\alpha)} = \frac{E_\alpha}{E^*}\bar{\sigma}_{11} \tag{17}$$

In this equation, E_α denotes the Young's modulus of the material within layer α and E^* is given by

$$E^* = v_f E_f + (1 - v_f)E_m \tag{18}$$

with E_f, E_m are the Young's moduli of the fiber and matrix materials, respectively, and v_f is the volume ratio of the fibers. Eq. (17) ensures that the axial strains in the 1-direction are equal in all layers. The critical buckling loads in the two different modes are determined as follows. For a given wave length Λ find the two consecutive roots of the determinant (in which replacement (16) has been performed). The correct critical buckling loads are those which correspond to the minimal values of Λ. As is discussed in the following, the first root of the determinant corresponds to the SBM while the second one provides the TBM. Consequently, the buckling of the layered composite takes place in the shear mode since TBM occurs under higher compressive load. This fact has been verified by employing the exact dispersion relation that has been established by Sun et al. (1968) for solely shear (anti-symmetric) wave propagation in periodically bilaminated composite (in which the repeating unit cell consists of two distinct layers). Thus, in terms of elastic wave propagation, the first root of the determinant corresponds to shear (anti-symmetric) wave propagation in periodically bilaminated composite. This implies that SBM can be directly obtained, in conjunction with (16), by utilizing the exact dispersion relation given by the 4th-order determinant that was established by Sun et al. (1968) for such types of wave. Indeed, it can be shown that our 20th-order determinant reduces to that of Sun et al. (1968) for solely shear waves propagating in periodically bilaminated composite. It should be mentioned that the dispersion relation in the latter case of periodic fiber and matrix layers has been also given by Brekhovskikh (1960) in the following compact form:

$$\frac{\mu_m \eta_m}{\mu_f \eta_f}\left[\tan^2\frac{\eta_f d_f}{2}+\tan^2\frac{\eta_m d_m}{2}\right]+\left[1+\left(\frac{\mu_m \eta_m}{\mu_f \eta_f}\right)^2\right]\tan\frac{\eta_f d_f}{2}\tan\frac{\eta_m d_m}{2}=0 \tag{19}$$

where d_f and d_m are the width of the fiber and matrix layers.

In Fig. 2, results are presented for $E_m = 3.64$ GPa, $E_f/E_m = 100$, $v_f = 0.2$ and $v_m = 0.35$, where v_f and v_m denote the Poisson's ratios of the fiber and matrix materials. This figure exhibits the critical wave length Λ, normalized with respect to the width of the fiber layer d_f, for various amounts of the fiber volume ratio $v_f = d_f/(d_f + d_m)$ where d_m is the width of the matrix layers, see Fig. 1(c). As mentioned above, the critical wave length are those at which the minimum values of the externally applied loading $\bar{\sigma}_{11}$, namely the buckling load, occur. Both SBM and TBM are shown, together with comparisons with the results obtained by the mechanics of materials approach of Parnes and Chiskis (2002). Good agreement between the exact and mechanics of materials approaches is shown to exist.

The corresponding values of the buckling strain $\bar{\epsilon}_{11} = \bar{\sigma}_{11}/E^*$ of the laminated composite that has been discussed in Fig. 2 are shown in Fig. 3 against v_f in both modes. It can be clearly observed that the critical

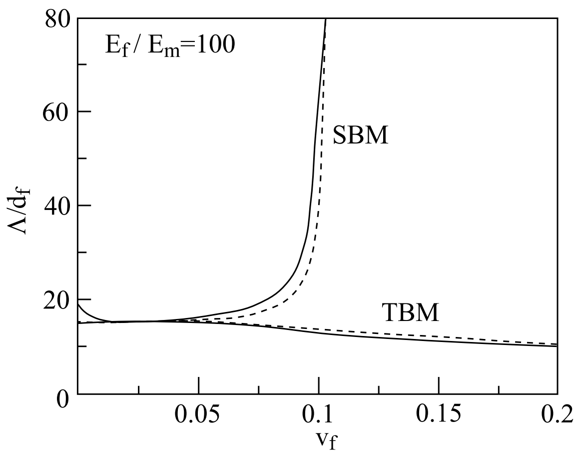

Fig. 2. Comparison between the exact (—) and the mechanics of material (- - -) prediction of the critical wave lengths at which buckling of the periodically layered composite occurs against fiber volume fraction. The material constants are: $E_m = 3.64$ GPa, $E_f/E_m = 100$, $v_f = 0.2$ and $v_m = 0.35$.

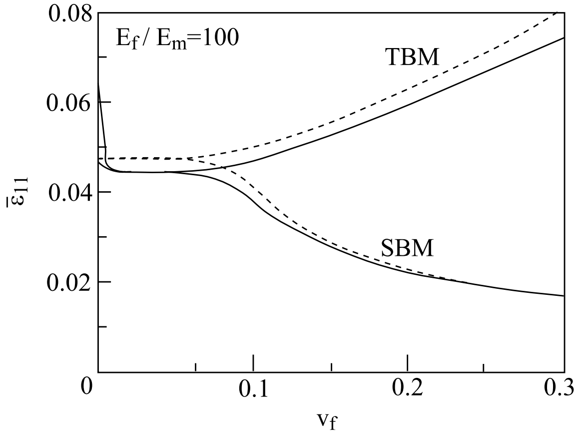

Fig. 3. Comparison between the exact (—) and the mechanics of material (---) prediction of the critical strains at which buckling of the periodically layered composite occurs against fiber volume fraction. The material constants are: $E_m = 3.64$ GPa, $E_f/E_m = 100$, $\nu_f = 0.2$ and $\nu_m = 0.35$.

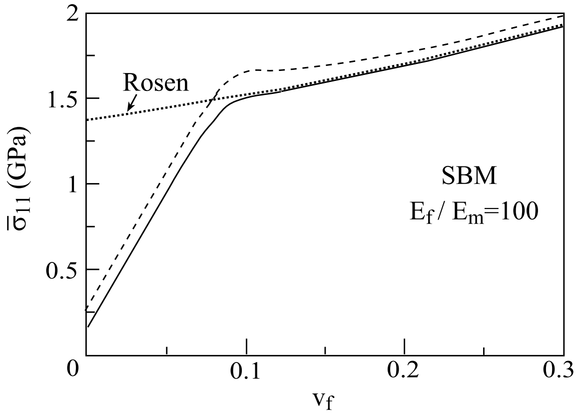

Fig. 4. Comparison between the exact (—) and the mechanics of material (---) prediction of the critical stress at which buckling in shear mode of the periodically layered composite occurs against fiber volume fraction. Also shown is the buckling stress provided by Rosen's long wave length limit which is given by: $\bar{\sigma}_{11} = \mu_m/(1 - v_f)$. The material constants are: $E_m = 3.64$ GPa, $E_f/E_m = 100$, $\nu_f = 0.2$ and $\nu_m = 0.35$.

values SBMs are always lower than the corresponding TBMs. Corresponding to Fig. 3 is Fig. 4 which shows the shear buckling stress against v_f. Also shown in this figure is the buckling stress in shear mode provided by Rosen's long wave length limit: $\bar{\sigma}_{11} = \mu_m/(1 - v_f)$ as well as the buckling stress as predicted by Parnes and Chiskis (2002) analysis. It is well seen that this limit forms an upper bound toward which the exact buckling stress approaches as v_f increases. The mechanics of materials based buckling stress of Parnes and Chiskis (2002), on the other hand, is seen to exceed this upper limit as the fiber volume fraction increases. This results from the approximations involved in the latter analysis. As mentioned by Parnes and Chiskis (2002), the deviation from Rosen's long wave length limit is significant at low values of v_f. This observation should be taken into account in the design of composites reinforced by nano-fibers where the fiber volume ratio is usually low.

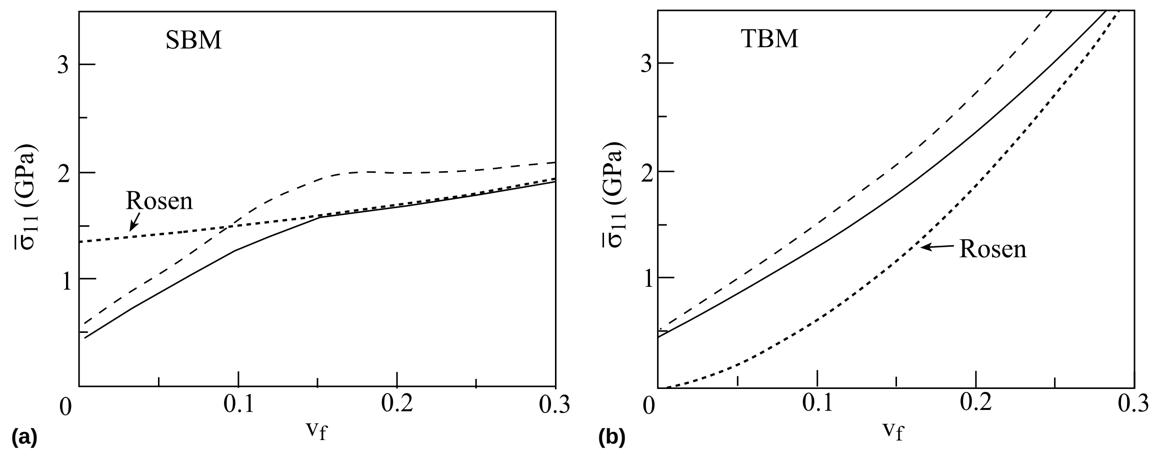

Fig. 5. Comparison between the exact (—) and the mechanics of material (---) prediction of the critical stress at which buckling in: (a) SBM and (b) TBM, of a glass/epoxy periodically layered composite occur against fiber volume fraction. Also shown are the Rosen's buckling stresses in shear and transverse modes. The material constants are: $E_f = 72.8$ GPa, $E_m = 3.64$ GPa, $\nu_f = 0.2$ and $\nu_m = 0.35$.

A final illustration of the prediction of the stress buckling is shown in Fig. 5 for a glass/epoxy periodically layered composite. Here $E_f = 72.8$ GPa, $\nu_f = 0.2$, $E_m = 36.4$ GPa and $\nu_m = 0.35$. The figure presents a comparison between the present exact and mechanics of materials approaches in SBM and TBM. Here too, Rosen's long wave limit forms an upper bound to the exact buckling load in shear while the mechanics of materials analysis exceeds this limit. Also shown is Rosen's transverse buckling mode which is given by: $2v_f[v_f E_f E_m/3(1-v_f)]^{1/2}$. It is clearly seen that for low values of volume fraction v_f this TBM prediction differs significantly from the present exact solution.

4. Buckling loads of continuous reinforced composites by wave propagation analogy

In the present section, the critical buckling loads in in-phase mode of long-fiber composites are predicted by employing the wave-buckling analogy. Fig. 6 shows a section of a continuous reinforced composite that is subjected to compressive stress loadings in the fibers direction whose average is $\bar{\sigma}_{11}$. In order to predict the SBM and the associated buckling-wave length Λ, a dispersion equation for harmonic shear wave propagation in the fiber direction (x_1-direction) of the composite is needed. We were able to detect three micromechanically based dispersion relations that provide the circular frequency ω of the shear waves for a given wave number $k = 2\pi/\Lambda$. The first dispersion equation is given by Achenbach and Herrmann (1968) which has been established in the first stage of development of their effective stiffness theory. It is given by

$$\left[\frac{\gamma(1+v_f)+1-v_f}{\gamma(1-v_f)+1+v_f}+\kappa\gamma v_f - v_f\frac{\rho_f\omega^2}{\mu_m k^2}-(1-v_f)\frac{\rho_m\omega^2}{\mu_m k^2}\right]\left[\frac{1+v_f}{2}\gamma(ka)^2+\kappa\gamma-\frac{\rho_f\omega^2(ka)^2}{4\mu_m k^2}\right]-v_f(\kappa\gamma)^2 = 0 \tag{20}$$

where a is the radius of the circular fiber, $\gamma = \mu_f/\mu_m$ and $\kappa = 0.847$ being a shear coefficient.

The second dispersion relation has been derived by Achenbach and Sun (1972) and is expressed as a 4th-order determinant whose elements are given by rather complicated expressions. Finally, the third dispersion relation has been presented by Achenbach (1976). It is based on an elaborate micromechanical analysis

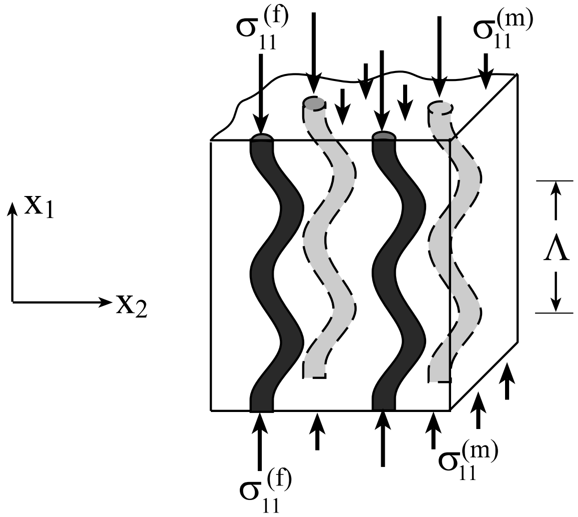

Fig. 6. Buckling of fibers in shear buckling mode in a continuous fiber composite. The fibers are oriented in the x_1-direction.

using an improved elastic field representation in the fiber and matrix regions of the repeating unit cell. It provides the following simple expression for the relationship between the frequency and wave number:

$$v_f \frac{\rho_f \omega^2}{k^2} + (1 - v_f)\frac{\rho_m \omega^2}{k^2} - a_1 + \frac{a_2^2}{a_4 k^2 - (v_f a^2 \rho_f \omega^2/4 + C\rho_m \omega^2) + a_3} = 0 \tag{21}$$

where

$$a_1 = v_f \mu_f + (1 - v_f)\mu_m, \quad a_2 = v_f(\mu_f - \mu_m), \quad a_3 = v_f \mu_f + \frac{v_f^2 \mu_m}{1 - v_f},$$
$$a_4 = 0.25 v_f(\lambda_f + 2\mu_f)a^2 + (\lambda_m + 2\mu_m)C$$

with

$$C = \left(\frac{4a^3}{3d} - \frac{3v_f a^2}{4}\right)\left(\frac{1}{1 - v_f}\right)^2 - \frac{1}{4}(d^2 - 3a^2)\left(\frac{v_f}{1 - v_f}\right)^2 + \frac{1}{4}\left(\frac{d^2}{3} - v_f a^2\right)\left(\frac{v_f}{1 - v_f}\right)^2$$

and $d = a\sqrt{\pi/v_f}$ being the size of the repeating square unit cell that consists of a circular fiber of radius a surrounded by the matrix material. In addition, Achenbach (1976) verified the latter dispersion relation by comparison with measured data showing satisfactory agreement. In Fig. 7, a comparison between the phase velocity $c = \omega/k$ and the wave number k as predicted by these three approximate dispersion relations is given. The materials constants are: $\mu_f/\mu_m = 100$, $\rho_f/\rho_m = 3$, $\nu_f = 0.3$, $\nu_m = 0.35$ and $v_f = 0.5$. It can be readily observed that the second and the third dispersion relation (Eq. (21)) are rather close to each other. Due to the simplicity of Eq. (21), it has been employed in conjunction with the wave-buckling analogy, Eqs. (16)–(18) in which α = f and m, in order to predict the critical buckling loads of the unidirectional long-fiber composite. It should be noted that a more accurate formula for computing the effective axial Young's modulus E^* of the unidirectional composite is given by (Christensen, 1979)

$$E^* = v_f E_f + (1 - v_f)E_m + \frac{4v_f(1 - v_f)(\nu_f - \nu_m)^2 \mu_m}{(1 - v_f)\mu_m/(\lambda_f + \mu_f) + v_f \mu_m/(\lambda_m + \mu_m) + 1}$$

In this way, the equality of the axial strain in the fiber and matrix phases is guaranteed.

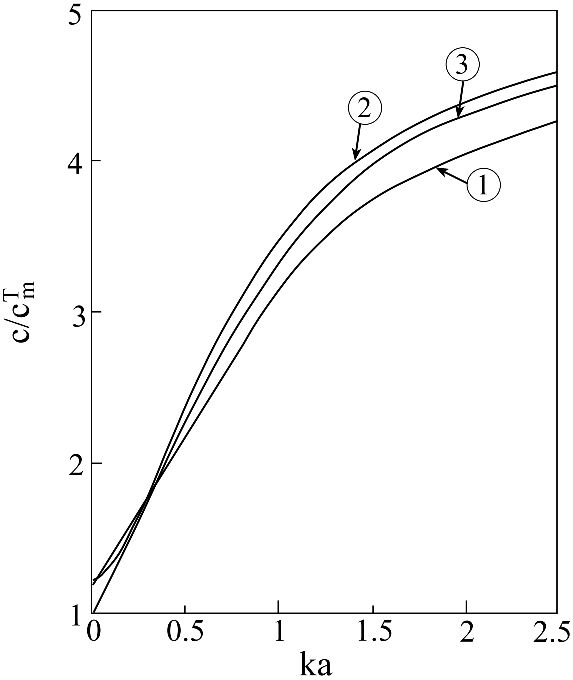

Fig. 7. Shear wave propagation in the fiber direction in a continuous fiber composite. Comparison between the phase velocity c, normalized with respect to the shear wave speed in the matrix $c_m^T = \sqrt{\mu_m/\rho_m}$, against the wave number k, normalized with respect to the fiber radius a. Curves 1–3 correspond to the dispersion relations obtained from Achenbach and Herrmann (1968) (Eq. (20)), Achenbach and Sun (1972) and Achenbach (1976) (Eq. (21)), respectively. The materials constants are: $\mu_f/\mu_m = 100$, $\rho_f/\rho_m = 3$, $\nu_f = 0.3$, $\nu_m = 0.35$ and $v_f = 0.5$.

Let us consider a long-fiber composite in which the fiber volume ratio v_f is sufficiently low to justify the categorization of the composite as dilute. For such a case, Sadowsky et al. (1967) presented a specific analysis for the prediction of the buckling loads of this type of unidirectional dilute composites. In the analysis of Sadowsky et al. (1967) the matrix is treated by the methods of elasticity whereas the single fiber is treated on the basis of the mechanics of materials approach. The two approaches are connected by the fiber–matrix interfacial conditions. In the final analysis these authors denoted the strain in the fiber by the (non-dimensional) parameter δ which in the present notation is given by

$$\delta = \frac{F}{\pi a^2 E_f} = \frac{\sigma_{11}^{(f)}}{E_f} = \frac{\bar{\sigma}_{11}}{E^*} \tag{22}$$

where F is the compression force within the fiber. Sadowsky et al. (1967) presented two types of results. In the first one, case 1, the fiber buckles into a sine curve but its cross section remains normal to the axis. In the second type of results, case 2, the cross section of the buckled fiber remains normal to the bent shape of the central line of the fiber. The latter case corresponds to the commonly used Euler–Bernoulli assumption. In Fig. 8, the two types of results (cases 1 and 2) of Sadowsky et al. (1967) are shown together with the buckling loads that are predicted on the basis of Eq. (21) in conjunction with (16) for various values of E_f/E_m with $v_f = 0.01$ (which ensures the dilute assumption). It can be observed that, in general, the buckling loads predicted on the basis of Eqs. (21) and (16) are higher than those given by the dilute analysis. In both analyses, however, $\delta \to 0$ as $E_f/E_m \to \infty$. This is expected since as the ratio E_f/E_m becomes higher, so are the

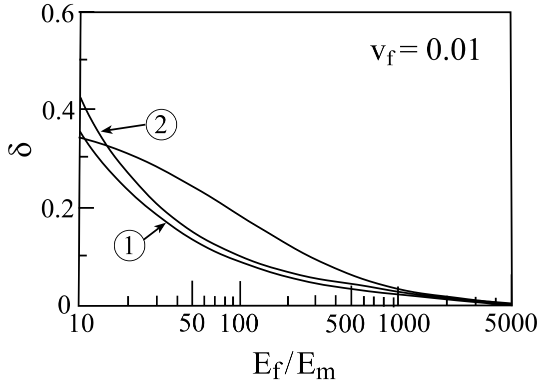

Fig. 8. A comparison between the buckling loads as predicted by cases 1 and 2 of Sadowsky et al. (1967) dilute composite and those based on Eq. (21) in conjunction with (16). The materials constants are: $\nu_f = 0.2$, $\nu_m = 0.4$ and the fiber volume ratio is $v_f = 0.01$.

corresponding buckling-wave lengths Λ. But $E_f/E_m \to \infty$ implies the gradual disappearance of the matrix yielding in such a case the Euler buckling formula

$$F = \frac{4\pi^2 E_f I}{\Lambda^2} \tag{23}$$

where $I = \pi a^2/4$ is the moment of inertia of the fiber. In conjunction with Eq. (22), this shows that δ becomes vanishingly small.

Consider next the non-dilute case of a continuous fiber composite in which $E_m = 3.64$ GPa, $\nu_m = 0.35$ and $\nu_f = 0.2$. Fig. 9 shows the buckling stresses $\bar{\sigma}_{11}$ against the fiber volume ration v_f for various values of fiber to matrix Young's moduli: E_f/E_m. This figure shows that significant differences occur at the higher

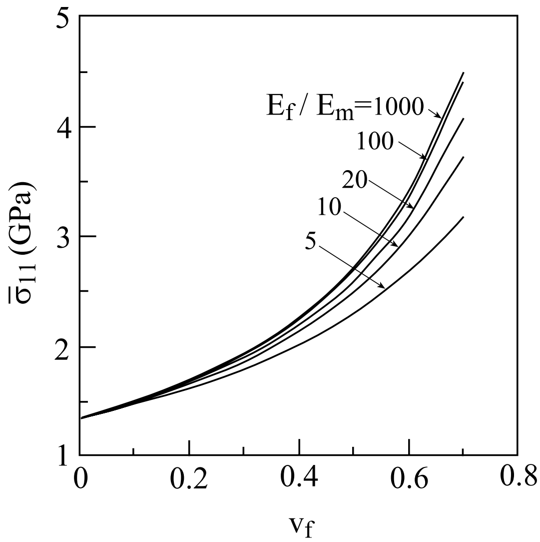

Fig. 9. Buckling loads of continuous fiber composites against the fiber volume ratio v_f for various values of fiber-to-matrix Young's moduli E_f/E_m. The material constants are: $E_m = 3.64$ GPa, $\nu_m = 0.35$ and $\nu_f = 0.2$. Rosen's long wave length limit, which is given by: $\bar{\sigma}_{11} = \mu_m/(1 - v_f)$, coincides with the curve $E_f/E_m = 1000$.

values of fiber volume ratios. For small values of v_f the critical buckling loads are close to each other. It should be noted that the dependence of the buckling loads on E_f/E_m in the dilute case of Fig. 8 was due to the specific normalization expressed by the parameter δ given by Eq. (22). It is interesting to mention that although Rosen's long wave length buckling load in shear mode: $\mu_m/(1-v_f)$ has been established by the analysis of periodic bilaminated material, it coincides in the present case of fiber-reinforced material with the curve of $E_f/E_m = 1000$ shown in Fig. 9 for all values of v_f. It can be concluded, therefore, that Rosen's expression is useful in such composites as long as the ratio between the fiber-to-matrix Young's moduli is sufficiently high.

The critical buckling loads of the continuous fiber composite are correspond to the minimal values of the applied load $\bar{\sigma}_{11}$ computed for various magnitudes of wave lengths Λ. These minimal values have been obtained in all cases shown in Fig. 9 as asymptotic values of Λ at which $\Lambda \to \infty$. This is illustrated in Fig. 10 for various values of v_f in the case in which the ratio of the fiber to matrix Young's moduli is: $E_f/E_m = 5$. The asymptotically determined critical buckling load $\bar{\sigma}_{11}$ can be readily obtained from Eq. (21), in conjunction with the replacement (16) and (17), in the limit of $\Lambda \to \infty$. It is given by the simple expression:

$$\bar{\sigma}_{11} = \frac{a_1 a_3 - a_2^2}{a_3[v_f E_f + (1 - v_f)E_m]} E^* \tag{24}$$

The initial value of $\bar{\sigma}_{11}$ at $\Lambda = 0$ can be also determined from Eq. (21) in the simple form:

$$\bar{\sigma}_{11} = a_1 \tag{25}$$

Obtaining the critical buckling loads $\bar{\sigma}_{11}$ from the asymptotic values of the wave lengths Λ in continuous fiber composites differs from the case of periodically layered composites. In the latter type of composites the minimal values of Λ are obtained as real minima at low fiber volume ratio v_f and as asymptotic values at higher values of v_f. This is illustrated in Fig. 11 which shows the variation of the applied loading $\bar{\sigma}_{11}$ against the wave length Λ for continuous fiber and periodically layered composites in both of which $E_f/E_m = 100$. For $v_f = 0.05$, Fig. 11(a), the critical buckling load $\bar{\sigma}_{11} = 0.96$ GPa of the layered composite is determined from the minimal value of $\Lambda/d_f = 15$, while for the continuous fiber composite the critical value

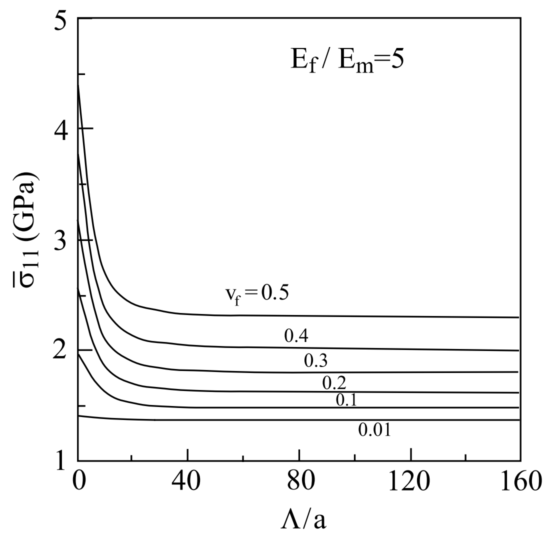

Fig. 10. The values of $\bar{\sigma}_{11}$ of continuous fiber composites against the wave lengths Λ/a. The material constants are: $E_m = 3.64$ GPa, $v_m = 0.35$, $E_f/E_m = 5$ and $v_f = 0.2$.

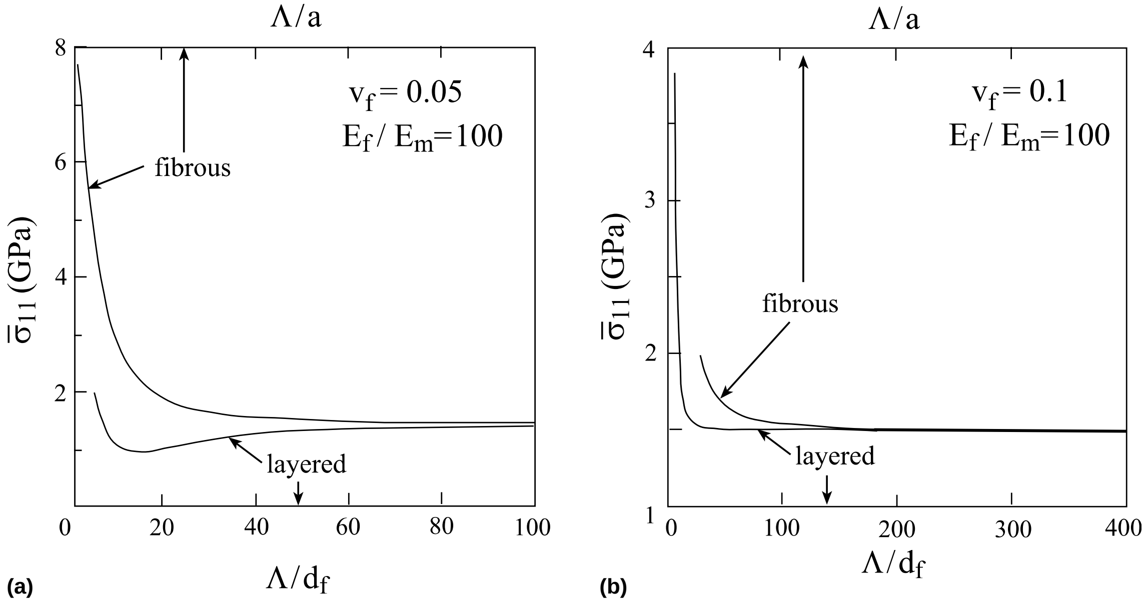

Fig. 11. The values of $\bar{\sigma}_{11}$ of periodic layered and continuous fiber composites against the wave lengths Λ. (a) Fiber volume ratio $v_f = 0.05$, and (b) fiber volume ratio $v_f = 0.1$. The material constants are: $E_m = 3.64$ GPa, $\nu_m = 0.35$, $E_f/E_m = 100$ and $\nu_f = 0.2$.

$\bar{\sigma}_{11} = 1.42$ GPa is determined from the asymptotic value of $\Lambda \to \infty$. On the other hand, for the higher value of the fiber volume ratio $v_f = 0.1$, Fig. 11(b), the minimal values of the wave lengths in both types of composites are obtained as asymptotic values. The corresponding buckling load in both cases is: $\bar{\sigma}_{11} = 1.5$ GPa.

5. Conclusions

The analogy between the equations of elastic wave propagation in solids and the buckling equations has been utilized to predict the buckling loads in periodic layered and continuous fiber composites. Due to the availability of exact dispersion relations for periodic layered materials and approximate dispersion relations for continuous fiber composites, it is possible to establish the corresponding exact and approximate buckling loads in these composites. The present approach can be extended to establish the critical buckling loads in composites with anisotropic layers, anisotropic fibers and in short-fiber composites. For periodically laminated composites with transversely isotropic layers, orthotropic layers and layers of monoclinic symmetries, exact dispersion relations have been presented by Nemat-Nasser and Yamada (1981), Yamada and Nemat-Nasser (1987) and Nayfeh (1991), respectively. Furthermore, it is possible to utilize the instantaneous properties of the inelastic phases in order to determine the corresponding buckling loads of inelastic composites.

Buckling of fibers in piezoelectric composites can be predicted by the present wave-buckling analogy depending on the availability of dispersion relations for propagating shear waves in such type of composites. For periodically layered piezoelectric composites, exact dispersion relations for harmonic wave propagation have been presented by Minagawa (1995) and Nayfeh et al. (1999). An adjustment of their analyses to propagating harmonic shear waves in the layering direction should provide the requested dispersion relation to be utilized in the corresponding buckling analysis. This is a topic for a future research since, to the authors knowledge, the subject of microbuckling of piezoelectric fibers has not as yet been investigated.

Acknowledgements

The authors wish to thank Dr. Steven M. Arnold, NASA Glenn Research Center, for his suggestion to investigate the buckling of fibers in composites. The first author gratefully acknowledges the support of the Diane and Arthur Belfer chair of Mechanics and Biomechanics.

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